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Compact composition operators on weighted Hilbert spaces of analytic functions ☆

Karim Kellay [a,1], Pascal Lefèvre [b,*]

[a] CMI, LATP, Universite de Provence, 39, rue F. Joliot-Curie, 13453 Marseille cedex 13, France
[b] Univ Lille Nord de France, U-Artois, Laboratoire de Mathématiques de Lens EA 2462, Fédération CNRS Nord-Pas-de-Calais FR 2956, F-62 300 Lens, France

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ABSTRACT

We characterize the compactness of composition operators acting on a large family of Hilbert spaces of analytic functions which lie between Bergman and Dirichlet spaces. Our characterization is given in terms of generalized Nevanlinna counting functions.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. Given a positive integrable function $\omega \in C^2[0,1)$, we extend it by $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, and call such ω a weight function. We denote by $\mathcal{H}_\omega$ the space of analytic functions f on $\mathbb{D}$ such that

$$\|f'\|_\omega^2 := \int_{\mathbb{D}} |f'(z)|^2 \omega(z)\, dA(z) < \infty,$$

where $dA(z) = dx\,dy/\pi$ stands for the normalized area measure in $\mathbb{D}$. The space $\mathcal{H}_\omega$ is endowed with the norm

$$\|f\|_{\mathcal{H}_\omega}^2 := |f(0)|^2 + \|f'\|_\omega^2.$$

A simple computation shows that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{H}_\omega$ if and only if

$$\|f\|_{\mathcal{H}_\omega}^2 = \sum_{n\geqslant 0} |a_n|^2 w_n < \infty,$$

where $w_0 = 1$ and

$$w_n = 2n^2 \int_0^1 r^{2n-1}\omega(r)\, dr, \quad n \geqslant 1.$$

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* Corresponding author.
E-mail addresses: kellay@cmi.univ-mrs.fr (K. Kellay), pascal.lefevre@euler.univ-artois.fr (P. Lefèvre).
[1] Current address: IMB, Universite Bordeaux I, 351 cours de la liberation, 33405 Talence cedex, France.

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Examples. Let $\alpha > -1$, $\omega_\alpha(r) = (1-r^2)^\alpha$ and denote $\mathcal{H}_{\omega_\alpha}$ by $\mathcal{H}_\alpha$. The Hardy space H^2 can be identified with $\mathcal{H}_1$. The Dirichlet space $\mathcal{D}_\alpha$ is precisely $\mathcal{H}_\alpha$ for $0 \leqslant \alpha < 1$ and $\mathcal{H}_0$ corresponds to the classical Dirichlet space $\mathcal{D}$. The Bergman spaces

$$\mathcal{A}^2_\alpha(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap \mathrm{L}^2\big(\mathbb{D}, \big(1-|z|^2\big)^\alpha \, dA(z)\big),$$

where $\alpha > -1$, can be identified with $\mathcal{H}_{\alpha+2}$.

In order to state our results, we introduce the notion of admissible weight.

Definition 1.1. A weight function ω is called admissible if

($\mathcal{W}_1$) ω is non-increasing,
($\mathcal{W}_2$) $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$,
($\mathcal{W}_3$) $\lim_{r\to 1-} \omega(r) = 0$.
($\mathcal{W}_4$) One of the two properties of convexity is fulfilled

$$\begin{cases} (\mathcal{W}_4^{(\mathrm{I})}): & \omega \text{ is convex and } \lim_{r\to 1} \omega'(r) = 0, \\ \text{or} & \\ (\mathcal{W}_4^{(\mathrm{II})}): & \omega \text{ is concave.} \end{cases}$$

Sometimes, we are going to be more specific: if ω satisfies conditions ($\mathcal{W}_1$)–($\mathcal{W}_3$) and ($\mathcal{W}_4^{(\mathrm{I})}$) (respectively ($\mathcal{W}_4^{(\mathrm{II})}$)), we shall say that ω is (I)-admissible (respectively (II)-admissible).

Examples. We point out that (I)-admissibility corresponds to the case $\mathrm{H}^2 \subsetneq \mathcal{H}_\omega \subset \mathcal{A}^2_\alpha(\mathbb{D})$ for some $\alpha > -1$, whereas (II)-admissibility corresponds to the case $\mathcal{D} \subsetneq \mathcal{H}_\omega \subseteq \mathrm{H}^2$. The weight $\omega_0 = 1$ is not an admissible weight, so the results of this paper do not apply to the Dirichlet space.

The Nevanlinna counting functions will play a key role in our work. See [5,6] for recent results on the classical Nevanlinna counting function and the quadratic Nevanlinna counting function.

Definition 1.2. Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. The generalized Nevanlinna counting function associated to ω is defined for every $z \in \mathbb{D} \setminus \{\varphi(0)\}$ by

$$N_{\varphi,\omega}(z) = \sum_{\substack{\varphi(a)=z \\ a\in\mathbb{D}}} \omega(a).$$

Note that $N_{\varphi,\omega}(z) = 0$ when $z \notin \varphi(\mathbb{D})$. By convention, we define $N_{\varphi,\omega}(z) = 0$ when $z = \varphi(0)$. When $\omega(r) = \omega_1(r) \sim \log 1/r$, $N_{\varphi,\omega_1} = N_\varphi$ is the usual Nevanlinna counting function associated to φ. The weighted Nevanlinna counting function was already considered in the special case of weighted Bergman spaces with standard weights (see [9] or [8] for instance).

In this note, we study composition operators on $\mathcal{H}_\omega$. The operator of composition with φ is defined as follows

$$C_\varphi(f) = f \circ \varphi, \quad \text{for } f \in \mathcal{H}_\omega.$$

The main result of the paper will concern compactness of C_φ. Nevertheless, before proving this result, we have to ensure the boundedness of C_φ. If φ is an holomorphic map on the unit disk $\mathbb{D}$ into itself, it is an easy consequence of Littlewood's subordination principle (see [2] or [10]) that C_φ is bounded on $\mathcal{H}_\omega$ for each (I)-admissible weight ω (see also Remark 2.6). For the case of (II)-admissible weights we have

Theorem 1.3. *Let ω be a (II)-admissible weight and $\varphi \in \mathcal{H}_\omega$. Then C_φ is bounded on $\mathcal{H}_\omega$ if and only if*

$$\sup_{|z|<1} \frac{N_{\varphi,\omega}(z)}{\omega(z)} < \infty. \tag{1}$$

The following theorem generalizes the previously known results of [9, Theorem 2.3, Corollary 6.11] or [3], on Hardy and Bergman spaces, see also Corollary 1.5.

Theorem 1.4. *Let ω be an admissible weight and $\varphi \in \mathcal{H}_\omega$. Then C_φ is compact on $\mathcal{H}_\omega$ if and only if*

$$\lim_{|z|\to 1^-} \frac{N_{\varphi,\omega}(z)}{\omega(z)} = 0. \tag{2}$$

Obviously, condition (2) implies the boundedness of C_φ on $\mathcal{H}_\omega$ for the admissible weight. Theorem 1.4 asserts that C_φ is compact on $\mathcal{D}_\alpha := \mathcal{H}_\alpha$ for $0 < \alpha < 1$ if and only if (2) is satisfied, i.e.

$$N_{\varphi,\alpha}(z) := \sum_{\substack{\varphi(w)=z \\ w\in\mathbb{D}}} \big(1-|w|^2\big)^\alpha = o\big(\big(1-|z|^2\big)^\alpha\big).$$

Note that $N_{\varphi,0}(z)$ is just the multiplicity $n_\varphi(z)$ of φ at z.

Let us recall that Zorboska showed in [11] (see also [3]) that, for $\varphi \in \mathcal{D}_\alpha$ where $0 \leqslant \alpha < 1$, C_φ is bounded on $\mathcal{D}_\alpha$ if and only if $N_{\varphi,\alpha}\,dA(z)$ is a Carleson measure for $\mathcal{A}_\alpha(\mathbb{D})$ and C_φ is compact on $\mathcal{D}_\alpha$ if and only if $N_{\varphi,\alpha}\,dA(z)$ is a vanishing Carleson measure for $\mathcal{A}_\alpha(\mathbb{D})$. More explicitly, for $0 \leqslant \alpha < 1$,

$$\begin{cases} C_\varphi \quad \text{is bounded on } \mathcal{D}_\alpha \iff \sup_{\zeta\in\mathbb{T}} \sup_{\delta>0} \dfrac{1}{\delta^{2+\alpha}} \displaystyle\int_{\{|z-\zeta|<\delta\}} N_{\varphi,\alpha}(z)\,dA(z) < \infty, \\ C_\varphi \quad \text{is compact on } \mathcal{D}_\alpha \iff \lim_{\delta\to 0} \dfrac{1}{\delta^{2+\alpha}} \sup_{\zeta\in\mathbb{T}} \displaystyle\int_{\{|z-\zeta|<\delta\}} N_{\varphi,\alpha}(z)\,dA(z) = 0. \end{cases}$$

We will recover these results for $\alpha > 0$ as simple consequences of our results (see Theorem 3.5).

There is another approach to the subject: given a continuous function $\sigma : [0,1) \to (0,\infty)$ such that $\sigma \in \mathrm{L}^1(0,1)$, we can consider the weighted Bergman space

$$\mathcal{A}^2_\sigma(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap \mathrm{L}^2(\mathbb{D}, \sigma\,dA)$$

consisting of analytic functions in $\mathbb{D}$ and square area integrable with respect to the weight σ. The space $\mathcal{A}^2_\sigma(\mathbb{D})$ is equipped with the norm

$$\|f\|_\sigma = \left(\int_{\mathbb{D}} \left|f(z)\right|^2 \sigma(z)\,dA(z)\right)^{1/2}.$$

If φ is a holomorphic map from the unit disk $\mathbb{D}$ into itself, by Littlewood's subordination principle, the composition operator C_φ is bounded on $\mathcal{A}^2_\sigma(\mathbb{D})$. A simple computation shows that a function $f(z) = \sum_{n=0}^\infty a_n z^n$ belongs to $\mathcal{A}^2_\sigma(\mathbb{D})$ if and only if

$$\|f\|^2_\sigma = \sum_{n\geqslant 0} |a_n|^2 \sigma_n < \infty,$$

where

$$\sigma_n = 2\int_0^1 r^{2n+1}\sigma(r)\,dr, \quad n \geqslant 0.$$

We associate to σ the weight given by

$$\omega_\sigma(r) = \int_r^1 (t-r)\sigma(t)\,dt.$$

We point out that $\lim_{r\to 1-} \omega'_\sigma(r) = 0$ since $\sigma \in \mathrm{L}^1(0,1)$ and that $\omega_\sigma''(r) = \sigma(r)$. We have

$$\frac{\sigma_{n+1}}{(1+n)^2} \asymp \int_0^1 r^{2n+1}\omega_\sigma(r)\,dr, \quad n \geqslant 0.$$

Therefore for every $f \in \mathcal{A}^2_\sigma(\mathbb{D})$, we have

$$\|f\|^2_\sigma \asymp \left|f(0)\right|^2 + \|f'\|^2_{\omega_\sigma},$$

so

$$\mathcal{A}^2_\sigma(\mathbb{D}) = \mathcal{H}_{\omega_\sigma}.$$

Moreover, it is worth pointing out that the weight ω_σ always satisfies $(\mathcal{W}_1)$, $(\mathcal{W}_3)$ and $(\mathcal{W}_4^{(\mathrm{I})})$. Thus ω_σ is (I)-admissible if and only if it satisfies $(\mathcal{W}_2)$. We have the following corollary.

Corollary 1.5. *Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Let σ be a weight such that ω_σ is* (I)*-admissible. Then C_φ is compact on $\mathcal{A}^2_\sigma(\mathbb{D})$ if and only if*

$$\lim_{|z|\to 1-} \frac{N_{\varphi,\omega_\sigma}(z)}{\omega_\sigma(z)} = 0.$$

Examples. Note that if $\alpha > -1$ and $\sigma_\alpha(r) = (1-r)^\alpha$, then $\omega_{\sigma_\alpha} = \omega_{\alpha+2}$. The composition operator C_φ is compact on $\mathcal{A}_\alpha(\mathbb{D}) = \mathcal{A}^2_{\sigma_\alpha}(\mathbb{D})$ if and only if

$$\lim_{|z|\to 1^-} \frac{1-|\varphi(z)|}{1-|z|} = \infty \tag{3}$$

(see [3,9]). Condition (3) means that φ does not have a finite angular derivative at any point of $\partial\mathbb{D}$. The compactness of C_φ on H^2 implies (3), but the angular derivative condition (2) is no longer sufficient for the compactness of C_φ on H^2 for the general case but still sufficient for finitely valent symbol (see [3]). Recall that φ is finitely valent when $\sup_{z\in\mathbb{D}} n_\varphi(z) < \infty$. We have the following corollary which involves a condition that can be viewed as a generalization of the condition (3).

Corollary 1.6. *Let σ be an admissible weight and $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$.*

- *If C_φ is compact on $\mathcal{H}_\omega$, then $\lim_{|z|\to 1^-} \frac{\omega_\sigma(z)}{\omega_\sigma(\varphi(z))} = 0$.*
- *The implication becomes an equivalence if φ is a finitely valent holomorphic function from the disk into itself.*

Another example where Corollary 1.5 applies is the following limiting case

$$\sigma(r) = \left(\left(1-r^2\right)\log\frac{e}{1-r^2}\log\log\frac{e_2}{1-r^2}\cdots\left(\log_p\frac{e_p}{1-r^2}\right)^2\right)^{-1},$$

where $\log_1 x = \log x$, $\log_{k+1} x = \log\log_k x$, $e_1 = e$ and $e_{k+1} = e^{e_k}$. For this weight, it is easy to see that

$$\omega_\sigma(r) \asymp \left(1-r^2\right)\left(\log_p\frac{e_p}{1-r^2}\right)^{-1} \quad \text{and} \quad \sigma_n \asymp 1/\log_p n$$

so that we are closer to the Hardy space than to any classical weighted Bergman space $\mathcal{A}_\alpha(\mathbb{D})$, where $\alpha > -1$.

In this paper, $f \asymp g$ means that there exist some constants α, $\beta > 0$ such that $\alpha f \leqslant g \leqslant \beta f$.

2. Proofs of Theorems 1.3 and 1.4

Let q_λ denote the automorphism of the unit disc given by

$$q_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}, \quad z \in \mathbb{D}.$$

Consider the function $\phi = q_{\varphi(0)} \circ \varphi$. Then $\phi : \mathbb{D} \to \mathbb{D}$ is analytic, $\phi(0) = 0$ and $C_{q_{\varphi(0)}}$ is bounded. Note that $C_\phi = C_\varphi C_{q_{\varphi(0)}}$ and since $\varphi = q_{\varphi(0)} \circ \phi$, we also have $C_\varphi = C_\phi C_{q_{\varphi(0)}}$. Therefore, C_φ is bounded if and only if C_ϕ is bounded. Also, C_φ is compact if and only if C_ϕ is compact. On the other hand, we have to verify that the same invariance occurs for Nevanlinna counting functions, but this is an easy consequence of the following remark

$$N_{q_{\varphi(0)}\circ\varphi,\omega}(z) = \sum_{q_{\varphi(0)}\circ\varphi(a)=z} \omega(a) = \sum_{\varphi(a)=q_{\varphi(0)}(z)} \omega(a) = N_{\varphi,\omega}\left(q_{\varphi(0)}(z)\right).$$

Finally, we can replace $\omega(q_{\varphi(0)}(z))$ by $\omega(z)$ in the conclusion thanks to the following remark.

Lemma 2.1. *If ω satisfies $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$ then there exists $C > 0$ such that*

$$\frac{1}{C}\omega(z) \leqslant \omega\left(q_{\varphi(0)}(z)\right) \leqslant C\omega(z), \quad z \in \mathbb{D}.$$

Proof. Set $q_{\varphi(0)}(z) = \zeta$ and suppose that $|\zeta| \geqslant |z|$. By $(\mathcal{W}_1)$, we have $\omega(\zeta) \leqslant \omega(z)$ and by $(\mathcal{W}_2)$ we get

$$\frac{\omega(z)}{\omega(\zeta)} = \frac{\omega(z)}{(1-|z|)^{1+\delta}}\frac{(1-|\zeta|)^{1+\delta}}{\omega(\zeta)}\frac{(1-|z|)^{1+\delta}}{(1-|\zeta|)^{1+\delta}} \leqslant 2^{1+\delta}\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1+\delta},$$

because

$$1-|\zeta|^2 = \frac{(1-|z|^2)(1-|\varphi(0)|^2)}{|1-\overline{\varphi(0)}z|^2}.$$

Finally, if $|\zeta| \leqslant |z|$, since $q_{\varphi(0)}(\zeta) = z$, it suffices to permute z and ζ in the former argument. $\square$

Hence, throughout the proof, we will assume that $\varphi(0) = 0$. We denote by $D(z,r)$ the disk of radius r centered at z. In order to prove the theorems, we shall need some lemmas

Lemma 2.2. *Let ω be a weight satisfying conditions $(\mathcal{W}_3)$ and $(\mathcal{W}_4^{(\mathrm{I})})$. Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$. Then the generalized Nevanlinna counting function $N_{\varphi,\omega}$ satisfies the sub-mean value property: for every $r > 0$ and every $z \in \mathbb{D}$ such that $D(z,r) \subset \mathbb{D} \setminus D(0,1/2)$*

$$N_{\varphi,\omega}(z) \leqslant \frac{2}{r^2} \int\limits_{D(z,r)} N_{\varphi,\omega}(\zeta)\, dA(\zeta).$$

Proof. We set $\frac{d^2\omega}{dt^2} = \sigma$, so

$$\omega(t) = \int\limits_t^1 (r-t)\sigma(r)\, dr.$$

Let $\varphi_r(z) = \varphi(rz)$, we have

$$N_{\varphi,\omega}(z) = \sum_{\varphi(\alpha)=z} \int\limits_{|\alpha|}^1 \big(r - |\alpha|\big)\sigma(r)\, dr = \int\limits_0^1 \sum_{\substack{\varphi(\alpha)=z\\ |\alpha|\leqslant r}} \big(r-|\alpha|\big)\sigma(r)\, dr.$$

Since $1/2 \leqslant |z| = |\varphi(\alpha)| \leqslant |\alpha| \leqslant r \leqslant 1$,

$$2\big(r-|\alpha|\big) \geqslant \log\big(r/|\alpha|\big) \geqslant r - |\alpha|.$$

So

$$2N_{\varphi,\omega}(z) \geqslant \int\limits_0^1 N_{\varphi_r}(z)\sigma(r)\, dr \geqslant N_{\varphi,\omega}(z). \tag{4}$$

So by (4), the generalized Nevanlinna counting function inherits the sub-mean value property from the classical Nevanlinna function (see [9] 4.6). □

Lemma 2.3. *Let ω be a (II)-admissible weight and let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$. Then the generalized Nevanlinna counting function $N_{\varphi,\omega}$ satisfies the sub-mean value property: for every $r > 0$ and every $z \in \mathbb{D}$ such that $D(z,r) \subset \mathbb{D} \setminus D(0,1/2)$*

$$N_{\varphi,\omega}(z) \leqslant \frac{2}{r^2} \int\limits_{D(z,r)} N_{\varphi,\omega}(\zeta)\, dA(\zeta).$$

Proof. By Aleman's formula [1, Lemma 2.3] for $\zeta, z \in \mathbb{D}$, let $\widetilde{q}_\zeta(z) = q_\zeta(-z)$ we have

$$N_{\varphi,\omega}(z) = -\frac{1}{2} \int\limits_{\mathbb{D}} \Delta\omega(\zeta) N_{f\circ\widetilde{q}_\zeta}(z)\, dA(\zeta).$$

Note that $\Delta\omega(\zeta) \leqslant 0$, since ω is decreasing and concave. We conclude as in the previous lemma. □

In order to prove the next lemma, we need the well-known estimate (see [4, Theorem 1.7]):

$$\int\limits_{\mathbb{D}} \frac{(1-|z|^2)^c\, dA(z)}{|1-z\bar{\lambda}|^{2+c+d}} \asymp \frac{1}{(1-|\lambda|^2)^d}, \quad \text{if } d > 0, c > -1. \tag{5}$$

Lemma 2.4. *Let ω be a weight satisfying $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$. Then*

$$\int\limits_{\mathbb{D}} \frac{\omega(z)\, dA(z)}{|1-\bar{\lambda}z|^{4+2\delta}} \asymp \frac{\omega(\lambda)}{(1-|\lambda|^2)^{2+2\delta}}.$$

Proof. Since ω is radial and non-increasing,

$$\int\limits_{|z|>\lambda} \frac{\omega(z)\, dA(z)}{|1-\bar{\lambda}z|^{4+2\delta}} \leqslant \omega(\lambda) \int\limits_{\mathbb{D}} \frac{dA(z)}{|1-\bar{\lambda}z|^{4+2\delta}} \asymp \frac{\omega(\lambda)}{(1-|\lambda|^2)^{2+2\delta}}.$$

The last equality follows from (5). On the other hand, since $\omega(r)/(1-r)^{1+\delta}$ is non-decreasing, we get

$$\int_{|z|<\lambda} \frac{\omega(z)}{(1-|z|^2)^{1+\delta}} \frac{(1-|z|^2)^{1+\delta}}{|1-\bar{\lambda}z|^{4+2\delta}}\,dA(z) \leqslant \frac{\omega(\lambda)}{(1-|\lambda|)^{1+\delta}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{1+\delta}\,dA(z)}{|1-\bar{\lambda}z|^{4+2\delta}} \asymp \frac{\omega(\lambda)}{(1-|\lambda|^2)^{1+\delta}} \frac{1}{(1-|\lambda|^2)^{1+\delta}}.$$

The last equality follows again from (5). The proof of the lower estimate is straightforward. □

Lemma 2.5. *Let ω be a weight satisfying $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$. For $\lambda \in \mathbb{D}$, set*

$$f_\lambda(z) = \frac{1}{\sqrt{\omega(\lambda)}} \frac{(1-|\lambda|^2)^{1+\delta}}{(1-\bar{\lambda}z)^{1+\delta}}.$$

Then

$$\|f_\lambda\|_{\mathcal{H}_\omega} \asymp 1.$$

Proof. On one hand, $f_\lambda(0) = \frac{(1-|\lambda|^2)^{1+\delta}}{\sqrt{\omega(\lambda)}}$ is bounded by $\frac{2^{1+\delta}}{\sqrt{\omega(0)}}$ thanks to (W_2). On the other hand,

$$\|f_\lambda'\|_\omega^2 \asymp \frac{(1-|\lambda|^2)^{2(1+\delta)}}{\omega(\lambda)} \int_{\mathbb{D}} \frac{\omega(z)}{|1-\bar{\lambda}z|^{4+2\delta}}\,dA(z).$$

The result follows then from Lemma 2.4. □

Proof of Theorem 1.3. Suppose that (1) is satisfied. The boundedness of C_φ follows from the change of variable formula [10]:

$$\|C_\varphi(f)\|_{\mathcal{H}_\omega}^2 = \bigl|f\bigl(\varphi(0)\bigr)\bigr|^2 + \int_{\mathbb{D}} \bigl|f'\bigl(\varphi(z)\bigr)\bigr|^2 \bigl|\varphi'(z)\bigr|^2 \omega(z)\,dA(z)$$
$$= \bigl|f(0)\bigr|^2 + \int_{\varphi(\mathbb{D})} \bigl|f'(z)\bigr|^2 N_{\varphi,\omega}(z)\,dA(z) \leqslant \bigl|f(0)\bigr|^2 + c\int_{\mathbb{D}} \bigl|f'(z)\bigr|^2 \omega(z)\,dA(z) \asymp \|f\|_{\mathcal{H}_\omega}^2.$$

Now assume that C_φ is bounded on $\mathcal{H}_\omega$. Let f_λ be the test function defined in Lemma 2.5. We have

$$\|C_\varphi \circ f_\lambda\|_{\mathcal{H}_\omega}^2 \asymp \frac{(1-|\lambda|^2)^{2+2\delta}}{\omega(\lambda)} \int_{\varphi(\mathbb{D})} \frac{N_{\varphi,\omega}(z)}{|1-\bar{\lambda}z|^{4+2\delta}}\,dA(z) \geqslant \frac{(1-|\lambda|^2)^{2+2\delta}}{\omega(\lambda)} \int_{D(\lambda,\frac{1-|\lambda|}{2})} \frac{N_{\varphi,\omega}(z)}{|1-\bar{\lambda}z|^{4+2\delta}}\,dA(z)$$
$$\geqslant c_1 \frac{1}{\omega(\lambda)} \frac{1}{(1-|\lambda|^2)^2} \int_{D(\lambda,\frac{1-|\lambda|}{2})} N_{\varphi,\omega}(z)\,dA(z) \geqslant c_2 \frac{N_{\varphi,\omega}(\lambda)}{\omega(\lambda)},$$

where the c_i's are independent of λ and the last inequality follows from Lemma 2.3. We conclude that

$$\sup_{\lambda\in\mathbb{D}} \frac{N_{\varphi,\omega}(\lambda)}{\omega(\lambda)} \leqslant c_3 \sup_{\lambda\in\mathbb{D}} \|C_\varphi \circ f_\lambda\|_{\mathcal{H}_\omega}^2 \leqslant c_3 \|C_\varphi\|^2 \sup_{\lambda\in\mathbb{D}} \|f_\lambda\|_{\mathcal{H}_\omega}^2,$$

which is bounded by virtue of Lemma 2.5 and the boundedness of C_φ on $\mathcal{H}_\omega$. □

Remark 2.6. Still assuming that $\varphi(0) = 0$, if ω is (I)-admissible, (1) is automatically satisfied. Indeed, the classical Littlewood inequality, applied to the function $r^{-1}\varphi_r$, gives that $N_{\varphi_r}(z) \leqslant \log(r/|z|)$ and so by (4)

$$N_{\varphi,\omega}(z) \leqslant \int_0^1 N_{\varphi_r}(z)\sigma(r)\,dr = \int_{|z|}^1 N_{\varphi_r}(z)\sigma(r)\,dr \leqslant \int_{|z|}^1 \log\bigl(r/|z|\bigr)\sigma(r)\,dr \leqslant 2\omega(z).$$

By Lemma 2.1, the same inequality holds without assuming $\varphi(0) = 0$.

Proof of Theorem 1.4. $\Longleftarrow$ Assume that (2) is satisfied. Let $(f_n)_n$ be a sequence in the unit ball of $\mathcal{H}_\omega$ converging weakly to 0. It suffices to show that $\|C_\varphi(f_n)\|_{\mathcal{H}_\omega} \to 0$ as $n \to \infty$. The weak convergence of f_n to 0 implies that $f_n(z) \to 0$ and $f_n'(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let $\varepsilon > 0$, there exists $\rho_\varepsilon \in (1/2, 1)$ such that

$$N_{\varphi,\omega}(z) \leqslant \varepsilon\omega(z), \qquad \rho_\varepsilon < |z| < 1.$$

By the change of variable formula

$$\begin{aligned}\left\|C_\varphi(f_n)\right\|^2_{\mathcal{H}_\omega} &\asymp \left|f_n(0)\right|^2+\left\|\varphi'\left(f_n'\circ\varphi\right)\right\|^2_\omega=\left|f_n(0)\right|^2+\int_{\mathbb{D}}\left|f_n'\big(\varphi(z)\big)\right|^2\left|\varphi'(z)\right|^2\omega(z)\,dA(z)\\
&=\left|f_n(0)\right|^2+\int_{\varphi(\mathbb{D})}\left|f_n'(z)\right|^2N_{\varphi,\omega}(z)\,dA(z)\leqslant\left|f_n(0)\right|^2+\int_{\rho_\varepsilon\mathbb{D}}\left|f_n'(z)\right|^2N_{\varphi,\omega}(z)\,dA(z)\\
&\quad+\varepsilon\int_{\varphi(\mathbb{D})\setminus\rho_\varepsilon\mathbb{D}}\left|f_n'(z)\right|^2\omega(z)\,dA(z)\leqslant\left|f_n(0)\right|^2+\int_{\rho_\varepsilon\mathbb{D}}\left|f_n'(z)\right|^2N_{\varphi,\omega}(z)\,dA(z)+\varepsilon.\end{aligned}$$

Since (f_n') converges uniformly to 0 on the closed disk $\rho_\epsilon\overline{\mathbb{D}}$, the conclusion follows easily.

$\Longrightarrow$ Let us assume that for a $\beta>0$ and a sequence $\lambda_n\in\mathbb{D}$ such that $|\lambda_n|\to 1^-$ we have

$$N_{\varphi,\omega}(\lambda_n)\geqslant\beta\omega(\lambda_n).$$

Let

$$f_n(z)=\frac{1}{\sqrt{\omega(\lambda_n)}}\frac{(1-|\lambda_n|^2)^{1+\delta}}{(1-\overline{\lambda_n}z)^{1+\delta}},\quad z\in\mathbb{D}.$$

By Lemma 2.5, $(f_n)_n$ is a bounded sequence on $\mathcal{H}_\omega$, converging weakly to 0. Indeed, it is uniformly converging to 0 on compact subsets since, by $(\mathcal{W}_2)$,

$$\frac{1}{\sqrt{\omega(\lambda_n)}}\left(1-|\lambda_n|^2\right)^{1+\delta}\leqslant\frac{2^{1+\delta}}{\sqrt{\omega(0)}}\left(1-|\lambda_n|\right)^{(1+\delta)/2}.$$

On the other hand, by the change of variable formula, Lemmas 2.2 and 2.3, we get

$$\begin{aligned}\left\|(f_n\circ\varphi)'\right\|^2_\omega &\asymp\frac{(1-|\lambda_n|^2)^{2+2\delta}}{\omega(\lambda_n)}\int_{\mathbb{D}}\frac{N_{\varphi,\omega}(z)}{|1-\overline{\lambda_n}z|^{4+2\delta}}\,dA(z)\geqslant\frac{c_1}{(1-|\lambda_n|^2)^2\omega(\lambda_n)}\int_{D(\lambda_n,\frac{1-|\lambda_n|}{2})}N_{\varphi,\omega}(z)\,dA(z)\\
&\geqslant c_1\frac{N_{\varphi,\omega}(\lambda_n)}{\omega(\lambda_n)}\geqslant c_2\beta,\end{aligned}$$

where c_1 and c_2 are positive constant independent of n. Thus C_φ cannot be compact, and this finishes the proof. □

3. Applications and complements

First let us indicate some special cases where Theorem 1.4 applies.

Proposition 3.1. 1) *Condition $(\mathcal{W}_2)$ is fulfilled for every classical weight $\sigma(r)=(1-r^2)^\alpha$ where $\alpha>-1$.*

2) *When σ is non-decreasing, then condition $(\mathcal{W}_2)$ is fulfilled by ω_σ with $\delta=1$.*

Proof. 1) Take $\delta=\alpha+1$.

2) We compute the derivative of $H(r)=\frac{\omega_\sigma(r)}{(1-r)^2}$:

$$H'(r)=\frac{2}{(1-r)^3}\int_r^1(x-\rho)\sigma(x)\,dx$$

where $\rho=(r+1)/2$. So

$$H'(r)=\frac{2}{(1-r)^3}\int_\rho^1(t-\rho)\big(\sigma(t)-\sigma(2\rho-t)\big)\,dt\geqslant 0$$

since σ is non-decreasing. □

It is known that the compactness on the Hardy space H^2 implies the compactness on classical weighted Bergman spaces. We are going to extend this result. On the other hand, it would be interesting to know when the non-angular derivative condition (3) is still equivalent to the compactness on weighted Bergman spaces. We also have a partial result in this direction. In order to state these results, we need some simple observations.

We associate to σ the weight ω_σ and we introduce the function

$$\Omega_\sigma(r) = \frac{\omega_\sigma(r)}{(1-r)}, \quad r \in [0, 1[$$

which is C^1 on $[0, 1[$ and extends continuously at 1 by

$$\Omega_\sigma(1) = -\lim_{r\to 1} \omega'_\sigma(r) = 0.$$

Moreover, when ω_σ verifies $(\mathcal{W}_3)$ and $\lim_{r\to 1} \omega'_\sigma(r) = 0$, we can write

$$\Omega_\sigma(r) = \int_r^1 \frac{\rho - r}{(1-r)} \sigma(\rho)\, d\rho,$$

so

$$\Omega'_\sigma(r) = \int_r^1 \frac{\rho - 1}{(1-r)^2} \sigma(\rho)\, d\rho.$$

Hence Ω_σ is a non-increasing function. Moreover,

$$\Omega''_\sigma(r) = \frac{\sigma(r)}{(1-r)} - \int_r^1 \frac{2(1-\rho)}{(1-r)^3} \sigma(\rho)\, d\rho = \frac{2}{(1-r)^3} \int_r^1 (1-\rho)\big(\sigma(r) - \sigma(\rho)\big)\, d\rho.$$

Therefore Ω_σ is a convex function when σ is a non-increasing function.

We write

$$\widetilde{\omega}_\sigma(x) = \omega_\sigma(1-x) \quad \text{and} \quad \widetilde{\Omega}_\sigma(x) = \Omega_\sigma(1-x).$$

The weight ω_σ is said to satisfy the condition (κ) if

$$\lim_{\eta\to 0^+} \limsup_{x\to 0^+} \frac{\tilde{\omega}_\sigma(\eta x)}{\eta \tilde{\omega}_\sigma(x)} = 0.$$

This is clearly equivalent to

$$\lim_{\eta\to 0^+} \limsup_{x\to 0^+} \frac{\widetilde{\Omega}_\sigma(\eta x)}{\widetilde{\Omega}_\sigma(x)} = 0.$$

Note that if σ is a non-increasing function, then ω_σ satisfies the condition (κ). Indeed, $\widetilde{\Omega}_\sigma(0) = 0$ and by convexity, $\widetilde{\Omega}_\sigma(\eta x) \leqslant \eta \widetilde{\Omega}_\sigma(x)$.

Theorem 3.2. *Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Let σ be a weight such that ω_σ is* (I)*-admissible.*

i) *The compactness of C_φ on the Hardy spaces H^2 always implies the compactness on $\mathcal{A}^2_\sigma(\mathbb{D})$.*
ii) *The compactness of C_φ on the weighted Bergman $\mathcal{A}^2_\sigma(\mathbb{D})$ always implies the compactness on the classical Bergman space $\mathcal{A}^2_0(\mathbb{D})$ (hence condition* (3) *is fulfilled).*

Proof. It suffices to treat the case $\varphi(0) = 0$.

i) The Schwarz lemma implies that for every $a \in \mathbb{D}$ with $\varphi(a) = z$, we have $|z| \leqslant |a|$. Hence $\Omega_\sigma(|z|) \geqslant \Omega_\sigma(|a|)$ and

$$N_{\varphi,\omega}(z) = \sum_{\substack{\varphi(a)=z\\ a\in\mathbb{D}}} \omega_\sigma(a) = \sum_{\substack{\varphi(a)=z\\ a\in\mathbb{D}}} \Omega_\sigma\big(|a|\big)\big(1-|a|\big) \leqslant \Omega_\sigma\big(|z|\big) \sum_{\substack{\varphi(a)=z\\ a\in\mathbb{D}}} \big(1-|a|\big) \leqslant \Omega_\sigma\big(|z|\big) o\big(1-|z|\big) = o\big(\omega_\sigma(z)\big).$$

ii) Using the function $\omega_\sigma(r)(1-r)^{-(1+\delta)}$ instead of Ω_σ, the same trick works to show that when C_φ is compact on $\mathcal{A}^2_\sigma(\mathbb{D})$, then C_φ is compact on $\mathcal{A}^2_{\sigma_{1+\delta}}(\mathbb{D})$. But this is known to be equivalent to the compactness on the standard Bergman space $\mathcal{A}^2_0(\mathbb{D})$. □

Now we are able to produce a simple sufficient test-condition to ensure that a composition operator on a weighted Bergman space is compact. We shall then see below that the converse of the first assertion in Theorem 3.2 is false, for any (I)-admissible weight.

Theorem 3.3. *Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Let σ be a weight such that ω_σ is* (I)*-admissible.*

i) *If*

$$\lim_{|z|\to 1^-} \frac{\Omega_\sigma(|z|)}{\Omega_\sigma(|\varphi(z)|)} = 0,$$

then C_φ is compact on $\mathcal{A}^2_\sigma(\mathbb{D})$.

ii) *When ω_σ satisfies the condition (κ), condition* (3) *implies that C_φ is compact on $\mathcal{A}^2_\sigma(\mathbb{D})$.*

Proof. It suffices to treat the case $\varphi(0) = 0$.

i) We fix $\varepsilon \in (0,1)$; there exists $\rho \in (0,1/2)$ such that for every $|a| > 1-\rho$, $\Omega_\sigma(a) \leqslant \varepsilon\Omega_\sigma(\varphi(a))$. By Schwarz's lemma, for every $a \in \mathbb{D}$ such that $\varphi(a) = z$, we have $|z| \leqslant |a|$. So if $|z| > 1-\rho$, then $|a| > 1-\rho$, where $\varphi(a) = z$. Hence for $z \in \mathbb{D}$ sufficiently close to the boundary of $\mathbb{D}$,

$$\begin{aligned} N_{\varphi,\omega_\sigma}(z) &= \sum_{\substack{\varphi(a)=z\\ a\in\mathbb{D}}} \Omega_\sigma\big(|a|\big)\big(1-|a|\big) \leqslant \varepsilon \sum_{\substack{\varphi(a)=z\\ a\in\mathbb{D}}} \Omega_\sigma\big(\big|\varphi(a)\big|\big)\big(1-|a|\big) \\ &= \varepsilon\Omega_\sigma\big(|z|\big) \sum_{\substack{\varphi(a)=z\\ a\in\mathbb{D}}} \big(1-|a|\big) \leqslant 2\varepsilon\Omega_\sigma\big(|z|\big)\big(1-|z|\big) \leqslant 2\varepsilon\omega_\sigma(z). \end{aligned}$$

ii) This is an immediate consequence of the preceding result. □

By Theorems 3.2 and 3.3, if

$$\frac{\omega_\sigma(z)}{\omega_\sigma(\varphi(z))} = o\left(\frac{1-|z|}{1-|\varphi(z)|}\right),$$

then C_φ is compact on $\mathcal{A}^2_\sigma(\mathbb{D})$ and

$$\frac{\omega_\sigma(z)}{\omega_\sigma(\varphi(z))} = o(1), \quad \text{when } |z| \to 1.$$

Of course, in the very special case of classical weighted Bergman spaces, we recover the well-known equivalence with condition (3).

We have already mentioned that the first implication in Theorem 3.2, is not an equivalence:

Corollary 3.4. *Let σ be a weight such that ω_σ is* (I)*-admissible. There exists an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ such that*

- *C_φ is compact on $\mathcal{A}^2_\sigma(\mathbb{D})$.*
- *C_φ is not compact on the classical Hardy space* H^2.

Proof. It suffices to apply both the preceding theorem and [7, Theorem 3.1] with

$$F(t) = \widetilde{\Omega}_\sigma^{-1}\big(\gamma\sqrt{\widetilde{\Omega}_\sigma(t)}\big),$$

where the numerical constant γ is chosen so that $F(1) = 1/2$. Note that F is non-decreasing and that $\lim_{t\to 0} F(t) = 0$. This provides us with a Blaschke product φ such that $\varphi(0) = 0$ and

$$1 - \big|\varphi(z)\big| \geqslant F\big(1-|z|\big).$$

Since φ is inner, C_φ cannot be compact on H^2. On the other hand,

$$\Omega_\sigma\big(\big|\varphi(z)\big|\big) = \widetilde{\Omega}_\sigma\big(1-\big|\varphi(z)\big|\big) \geqslant \widetilde{\Omega}_\sigma\big(F\big(1-|z|\big)\big) = \gamma\sqrt{\Omega_\sigma\big(|z|\big)}.$$

Hence

$$\lim_{|z|\to 1^-} \frac{\Omega_\sigma(|z|)}{\Omega_\sigma(|\varphi(z)|)} \leqslant \gamma^{-1} \lim_{|z|\to 1^-} \sqrt{\Omega_\sigma\big(|z|\big)} = 0. \qquad \square$$

As stated in the introduction, we can recover the characterization due to Zorboska of compactness for the weighted Dirichlet spaces.

Theorem 3.5. *Let ω be a* (II)*-admissible weight and let $\varphi \in \mathcal{H}_\omega$. Then*

(i) *C_φ is bounded on $\mathcal{H}_\omega$ if and only if*

$$\sup_{\delta>0}\sup_{\zeta\in\mathbb{T}} \frac{1}{\delta^2\omega(1-\delta)} \int\limits_{\{z\in\mathbb{D}:\ |z-\zeta|<\delta\}} N_{\varphi,\omega}(z)\,dA(z) < \infty.$$

(ii) *C_φ is compact on $\mathcal{H}_\omega$ if and only if*

$$\lim_{\delta\to 0}\sup_{\zeta\in\mathbb{T}} \frac{1}{\delta^2\omega(1-\delta)} \int\limits_{\{z\in\mathbb{D}:\ |z-\zeta|<\delta\}} N_{\varphi,\omega}(z)\,dA(z) = 0.$$

Proof. We prove only (ii) since the proof of (i) is similar. If we assume that C_φ is compact, then characterization (2) easily implies that

$$\lim_{\delta\to 0} \frac{1}{\delta^2\omega(1-\delta)} \int\limits_{\{z\in\mathbb{D}:\ |z-\zeta|<\delta\}} N_{\varphi,\omega}(z)\,dA(z) = 0$$

uniformly in $\zeta \in \mathbb{T}$.

Conversely, let $\eta \in \mathbb{D}$ such that $|\eta| > 1/2$. Let $\zeta = \eta/|\eta| \in \mathbb{T}$ and $\delta > 0$ such that η is the midpoint of $[(1-\delta)\zeta, \zeta]$. Hence, $\delta = 2(1-|\eta|) \in (0,1)$. Then by Lemma 2.2 and $(\mathcal{W}_4^{(\mathrm{II})})$, we get

$$\frac{1}{\delta^2\omega(1-\delta)} \int\limits_{\{z\in\mathbb{D}:\ |z-\zeta|<\delta\}} N_{\varphi,\omega}(z)\,dA(z) \geqslant \frac{1}{\delta^2\omega(1-\delta)} \int\limits_{\{z\in\mathbb{D}:\ |z-\eta|<\delta/2\}} N_{\varphi,\omega}(z)\,dA(z) \geqslant c\frac{N_{\varphi,\omega}(\eta)}{\omega(\eta)}.$$

Letting $|\eta| \to 1$, we get characterization (2). □

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